Unified FPT framework for crossing number problems

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The basic (and traditional) *crossing number problem* is to determine the minimum number of crossings in a topological drawing of an input graph in the plane. We develop a unified framework yielding fixed-parameter tractable (FPT) algorithms for many generalized crossing number problems not only in the plane.

Our framework takes the following form.

- We fix a surface Σ and a class $\mathcal{D}$ of "allowed" drawings of graphs in Σ (e.g., a class of drawings with at most t crossings and with additional constraints).
- We assume that testing membership in $\mathcal{D}$ can be done algorithmically (not necessarily in polynomial time), and that restricting a drawing in $\mathcal{D}$ to a subgraph, extending it without adding any crossing, and transforming it with a self-homeomorphism of Σ yields a drawing that is again in $\mathcal{D}$.

Then deciding whether an input graph G has a drawing in $\mathcal{D}$, and computing one if it is the case, is fixed-parameter tractable in (essentially) the genus of Σ and the maximum number of crossings in a drawing in $\mathcal{D}$.

More generally, we may take as input an edge-colored graph and distinguish crossings by the colors of the involved edges, we may count the crossings differently than "one by one", and we may allow a bounded number of colored edge removals and vertex splits on G before drawing it.

The proof is a reduction to the embeddability of a graph on a two-dimensional simplicial complex [1].

Our framework implies, in a unified way, linear or quadratic FPT algorithms for many topological crossing number variants established in the graph drawing community. Some of these variants already had previously published FPT algorithms, mostly relying on Courcelle's metatheorem such as [3, 2], but our algorithms have a better runtime. Moreover, our framework extends to these crossing number variants in any fixed surface, and also allows to fix the rotation system of the drawing of a graph in some variants.

References

[1] É. Colin de Verdière, T. Magnard, An FPT algorithm for the embeddability of graphs into two-dimensional simplicial complexes, SIAM J. Comput. 55(2) (2026) 410–450.

[2] T. Hamm, F. Klute, I. Parada, Computing crossing numbers with topological and geometric restrictions, arXiv:2412.13092, 2024.
[3] M. Münch, I. Rutter, Parameterized algorithms for beyond-planar crossing numbers, 32nd International Symposium on Graph Drawing and Network Visualization (GD 2024), LIPIcs 320 (2024) 25:1–25:16.