Colour-critical graphs and topological combinatorics

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(joint work with Matěj Stehlík, and partly with Carmen Arana and Penny Haxell)

With the seminal result of Lovász [5] that established the chromatic number of Kneser graphs, topological methods based on variants and combinatorial counterparts of the Borsuk–Ulam theorem have entered the realm of graph colouring. Early extensions of Lovász' result include Schrijver's [6] discovery of vertex-critical subgraphs of Kneser graphs with the same chromatic number, now called Schrijver graphs.

In this talk, I will give an overview of the results of a long-term joint project with Matěj Stehlík, initiated in [2] and presenting a geometric approach to the above results based on the concept of a quadrangulation of a space. It has turned out that other notions, such as the generalised Mycielski construction, fit this framework very naturally.

A combination of geometric and combinatorial reasoning, together with computer experiments, has led to one outcome of the project I will discuss in particular, namely an explicit description of *edge-critical* subgraphs of Schrijver graphs with the same chromatic number [3, 4]. This can be seen as a natural further step in the direction taken by Schrijver in [6].

In a more general context, Schrijver graphs belong to the class of generalised Kneser graphs (whose vertex set consists of stable sets of size k in a base graph) studied in [1] for $k = 2$. I will present partial results of ongoing research (joint with Carmen Arana, Penny Haxell and Matěj Stehlík) that aims to extend the results of [1] to arbitrary k.

References

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