Conflict-free coloring of planar graphs with four colors

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(joint work with Petr Hliněný)

We show that every planar graph admits an open-neighborhood conflict-free coloring with at most 4 colors. In such a coloring, every vertex is assigned a color and has a neighbor whose color is unique within its open-neighborhood.

Earlier work of Abel et al. [1] established a general bound of 9 colors for planar graphs and asked whether this bound can be improved. Huang et al. [2] later reduced the bound to 5 colors. More recently, Fabrici et al. [3] formulated the conjecture that 4 colors always suffice for planar graphs. Our result confirms this conjecture, and the bound is best possible.

Our approach combines structural properties of planar graphs with matching theory. Starting from a maximum matching, we analyze the structure around vertices that are not incident to any edge of the matching using a refined form of the Gallai–Edmonds decomposition. A key ingredient is a new structural result showing that every planar graph admits a maximum matching in which all such vertices have degree at most 5. This reduction allows us to focus on a small number of local configurations, which can either be simplified or shown to enforce the presence of specific substructures incompatible with planarity.

This yields a constructive framework that transforms the problem into a proper coloring of a derived planar graph, leading to the desired bound. In fact, our result can be viewed as equivalent to the Four Color Theorem.

References

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[3] I. Fabrici, B. Lužar, S. Rindošová, R. Soták, Proper conflict-free and unique-maximum colorings of planar graphs with respect to neighborhoods, Discrete Appl. Math. 324 (2023) 80–92.