Min cut-path problem

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Secure communication in distributed systems often requires the establishment of a trusted routing path while simultaneously disabling alternative routes accessible to adversaries. We formalize this dual objective of connectivity and isolation via the *cut-path* a hybrid graph structure comprising a set of edges that contains both a connecting path and a separating cut between two distinguished vertices u and v. The corresponding minimization problem we denote by Min Cut-Path.

We show that the decision version of Min Cut-Path is NP-complete. The proof employs a two-step polynomial-time reduction from 3-Sat [1]. To bridge the gap, we introduce an intermediate problem, Separating Shortest Path, which asks whether there exists a shortest path whose removal disconnects the target vertices. Using a modular construction of "chain" and "thread" gadgets to simulate the logical structure of variables and clauses, we establish the NP-hardness of this intermediate problem, which subsequently proves the NP-completeness of Min Cut-Path.

Despite its general hardness, we identify critical structural classes where the problem is polynomially solvable: graphs with a diameter of two [2] and graphs with a minimum *cut-value* at most two (e.g., cactus representations [3, 4]). Using a topological parity property that the intersection of any u-v cut and u-v path must contain an odd number of edges—we derive a closed form solution for these regimes. Specifically, we prove that the exact minimum *cut-path* value is uniquely determined by the minimum cut and shortest path distances: $cp(u, v) = c(u, v) + d(u, v) - 1$.

References

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