Extremal asymmetric depth of planar and higher-genus graphs

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Although almost all graphs are known to be asymmetric — having no nontrivial global automorphisms — they may still possess local symmetries in the form of isomorphisms between induced subgraphs, i.e., partial automorphisms. We study such local symmetries via *asymmetric depth*, defined as $d(\mathit{\Gamma}) = n - k_{\max}(\mathit{\Gamma})$, where $k_{\max}(\mathit{\Gamma})$ is the maximum rank of a nontrivial partial automorphism of an n-vertex graph $\mathit{\Gamma}$.

Continuing this line of work on specific graph classes, we prove a tight upper bound $d(\mathit{\Gamma}) \leq 5$ in the class of planar graphs and show that duals of IPR fullerenes can attain this extremal value: any fullerene whose dual achieves $d = 5$ must satisfy the *isolated pentagon rule*, the smallest examples occurring on 90 and 92 vertices. Exhaustive computations up to $n = 120$ indicate that almost all IPR fullerenes are asymmetric with depth 4, and those that fail to attain the maximum exhibit hidden, almost-global symmetries broken only locally. We extend the bound to surfaces of higher genus, obtaining $d(\mathit{\Gamma}) \leq 5 + \sqrt{1 + 48g}$.

Paralleling Frucht's gadget technique, we construct an explicit family $\{A_h\}_{h \geq 2}$ by attaching columns of varying lengths of h-dimensional hypercubes to a central copy of Q_h, and prove $d(A_h) = 2(h-2)$ for $h \geq 3$. Combined with a depth-monotonicity result for induced subgraphs, this realizes every positive integer as an asymmetric depth with increasingly high genus.

References

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