Improper coloring of graphs

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Improper coloring can be defined as a vertex decomposition of a graph into induced subgraphs with prescribed structural properties. This definition covers also classical results for planar graphs, such as decomposition into two outerplanar graphs and into three forests in general (arboricity 3) or into two forests in the triangle-free case. In this contribution we will mainly focus on the case where the induced subgraphs have bounded degree. Here we denote the corresponding coloring as a $(d_1, \ldots, d_k)$-coloring. Hence, for $d_1 = \cdots = d_k = 0$ we obtain a definition of a proper k-coloring. The most famous results include the $(0, 0, 0, 0)$- and $(2, 2, 2)$-colorability of planar graphs.

For graphs on surfaces, Choi and Esperet [1] obtained general upper bounds for improper colorings depending on the Euler genus. On torus every graph is properly 7-colorable, and as K_7 is embedable on the torus, the bound is tight. If we require to use less colors, some defect has to be non zero. The main question is what is the optimal value for individual defects.

For the torus we prove the existence of improper colorings with small defects. In particular, every toroidal graph admits a $(0, 0, 0, 3)$- and $(0, 0, 2, 2)$-coloring. We also present results for five and six colors based on Thomassen's characterization of 5-colorable toroidal graphs [2].

These results substantially strengthen the genus 1 bounds of Choi–Esperet. They show that, analogously as in planar case, toroidal graphs admit nearly optimal improper colorings when the number of colors is below the proper chromatic threshold.

References

[1] I. Choi, L. Esperet, Improper coloring of graphs on surfaces, J. Graph Theory 91(1) (2019) 16–34.

[2] C. Thomassen, Five-coloring graphs on the torus, J. Combin. Theory Ser. B 62(1) (1994) 11–33.